The Relation Between the Kronecker Product, the Trace Calculation, and the One-Dimensional Ising Model

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ABSTRACT

An identity is proved which is important for the trace calculation of a certain class of Kronecker products of matrices.

In the present paper we prove an identity concerning the trace of the exponential function of a certain class of Kronecker products of matrices.

First of all let us introduce the notation. I_n denotes the $n\times n$ unit matrix, and σ_z the 2×2 Pauli matrix

$$\sigma_z=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

$A\otimes B$ denotes the Kronecker product (*Tensorprodukt*) of the matrices A and B, and $\operatorname{tr} C$ the trace of the $n\times n$ matrix C. $[D, E]$ denotes the commutator (Lie bracket) of the $n\times n$ matrices D and E. Now let (I is the unit 2×2 matrix)

$$X_{12}\equiv\sigma_z\otimes\sigma_z\otimes I\otimes I,$$

$$X_{23}\equiv I\otimes\sigma_z\otimes\sigma_z\otimes I,$$

$$X_{34}\equiv I\otimes I\otimes\sigma_z\otimes\sigma_z,$$

$$X_{41}\equiv\sigma_z\otimes I\otimes I\otimes\sigma_z,$$

and

$$M=\{X_{12}, X_{23}, X_{34}, X_{41}\}.$$

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Thus the elements of M are $2^4 \times 2^4$ matrices. Finally we recall the identities

$$\operatorname{tr}(A+B) \equiv \operatorname{tr} A + \operatorname{tr} B$$

$$\operatorname{tr}(A \otimes B) \equiv (\operatorname{tr} A)(\operatorname{tr} B),$$

where A and B are $n \times n$ matrices.

In order to prove the main theorem of this paper, we need the following lemmata:

LEMMA 1. *Let X be a real $n \times n$ matrix such that $X^2 = I_n$. Let $\lambda \in \mathbb{R}$. Then*

$$\exp(\lambda X) \equiv I_n \cosh \lambda + X \sinh \lambda .$$

REMARK. The matrices given above are examples of such matrices.

LEMMA 2. *Let X and Y be arbitrary elements of M. Then*

$$[X, Y] = 0.$$

LEMMA 3. *Let X, Y, and Z be arbitrary elements of M. Then*

(i) $\operatorname{tr} X = 0$,
(ii) $\operatorname{tr}(XY) = 0$, *if* $X \neq Y$,
(iii) $\operatorname{tr}(XYZ) = 0$ *if* $X \neq Y$.

LEMMA 4. *Let $X_{12}, X_{23}, X_{34}, X_{41}$ be the matrices given above. Then*

(i) $X_{12} X_{23} X_{34} X_{41} = I_{16}$,
(ii) $\operatorname{tr}(X_{12} X_{23} X_{34} X_{41}) = 2^4 = 16$.

Now we are able to prove the following theorem:

THEOREM 1. *Let*

$$K \equiv \lambda (X_{12} + X_{23} + X_{34} + X_{41}).$$

Then

$$\operatorname{tr} \exp(K) \equiv \operatorname{tr}\left(I_{16} \cosh^4 \lambda + I_{16} \sinh^4 \lambda\right). \tag{1}$$

Proof. Using the lemmata given above we find

$$\begin{aligned}
\operatorname{tr}\exp(K) &\equiv \operatorname{tr}\left[\exp(\lambda X_{12})\exp(\lambda X_{23})\exp(\lambda X_{34})\exp(\lambda X_{41})\right]\\
&= \operatorname{tr}\left[(I_{16}\cosh\lambda + X_{12}\sinh\lambda)(I_{16}\cosh\lambda + X_{23}\sinh\lambda)\right.\\
&\qquad \left.\times(I_{16}\cosh\lambda + X_{34}\sinh\lambda)(I_{16}\cosh\lambda + X_{41}\sinh\lambda)\right]\\
&= \operatorname{tr}\left\{\left[I_{16}\cosh^2\lambda + (X_{12}+X_{23})\cosh\lambda\sinh\lambda + X_{12}X_{23}\sinh^2\lambda\right]\right.\\
&\qquad \left.\times\left[I_{16}\cosh^2\lambda + (X_{34}+X_{41})\cosh\lambda\sinh\lambda + X_{34}X_{41}\sinh^2\lambda\right]\right\}\\
&= \operatorname{tr}\left[I_{16}\cosh^4\lambda + I_{16}\sinh^4\lambda\right].
\end{aligned}$$

As a consequence we obtain

$$\operatorname{tr}\left(I_{16}\cosh^4\lambda + I_{16}\sinh^4\lambda\right) \equiv 2^4(\cosh^4\lambda + \sinh^4\lambda). \qquad \blacksquare$$

REMARK. In physics we call $\operatorname{tr}\exp(K)$ the partition function.

The given theorem can be extended to arbitrary dimensions.

THEOREM 2. *Let*

$$X_{12} \equiv \underbrace{\sigma_z \otimes \sigma_z \otimes I \otimes \cdots \otimes I}_{N\ \text{factors}},$$

$$X_{23} \equiv I \otimes \sigma_z \otimes \sigma_z \otimes \cdots \otimes I,$$

$$\vdots$$

$$X_{N-1,N} \equiv I \otimes I \otimes \cdots \otimes I \otimes \sigma_z \otimes \sigma_z,$$

$$X_{N1} \equiv \sigma_z \otimes I \otimes \cdots \otimes I \otimes \sigma_z,$$

where I is the unit 2×2 matrix. Let

$$K = \lambda(X_{12} + X_{23} + \cdots + X_{N-1,N} + X_{N1}),$$

where $\lambda \in \mathbb{R}$. *Then*

$$\operatorname{tr}\exp(K) \equiv \operatorname{tr}\left(I_N \cosh^N \lambda + I_N \sinh^N \lambda\right), \tag{2}$$

where I_N *is the unit* $2^N \times 2^N$ *matrix.*

As a consequence we obtain

$$\operatorname{tr}\exp(K) \equiv 2^N \cosh^N \lambda + 2^N \sinh^N \lambda. \tag{3}$$

REMARK 1. The lemmata described above can easily be extended to arbitrary dimensions. Consequently, the proof of Theorem 2 can be performed in the same way as the proof of the Theorem 1.

REMARK 2. When we take the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

or

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

instead of σ_z, we obtain the same result [Eq. (2)], because $\sigma_x^2 = \sigma_y^2 = I$.

REMARK 3. The $2^N \times 2^N$ matrix K describes the one-dimensional Ising model with cyclic boundary conditions. Consequently, the right-hand side of Eq. (3) is the partition function of the one-dimensional Ising model without an external field. The given result can be found in textbooks (for example Huang [1], Callaway [2]), but the present approach is quite different from those given in the textbooks. For the sake of completeness let us briefly describe the traditional approach. In one dimension the Ising model is given by

$$H_N = -J \sum_{i=1}^{N} \sigma_i \sigma_{i+1},$$

where $\sigma_{N+1} = \sigma_1$ (cyclic boundary conditions). N is the number of lattice sites, and J ($J > 0$) is a real constant. σ_i ($i = 1, \ldots, N$) is restricted to have two values ± 1. The partition function is

$$Z_N(\beta J) = \sum_{\sigma_1, \ldots, \sigma_N \in \Omega} \exp\left(\beta J \sum_{i=1}^{N} \sigma_i \sigma_{i+1} \right),$$

where the first sum runs over all possible configurations Ω. There are 2^N configurations. Hence, we must sum over all possible configurations, namely 2^N, instead of calculating the trace of a $2^N \times 2^N$ matrix.

REFERENCES

1 K. Huang, *Statistical Mechanics*, Wiley, New York, 1964.

2. J. Callaway, *Quantum Theory of the Solid State*, Part A, Academic, New York, 1974.

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